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1985 J. Phys. A: Math. Gen. 18 L829

(http://iopscience.iop.org/0305-4470/18/14/001)

J. Phys. A: Math. Gen. 18 (1985) L829-L832. Printed in Great Britain

LETTER TO THE EDITOR

On the 'quaternionic' bifurcation

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Received 15 July 1985

Abstract. As a generalisation of the classical Hopf bifurcation problem, which naturally exhibits an intrinsic S^1 symmetry, we propose an explicit example of bifurcation, covariant with respect to the group SU_2 and with a peculiar group theoretical feature, described by a 'quaternionic-type' representation.

As stressed by Golubitsky (1983) and by Golubitsky and Stewart (1985), bifurcation problems in the presence of a symmetry group G, when the symmetry is described by a real irreducible representation of G, fall into three structurally different cases. In fact, according to the classical Schur lemma (see Kirillov 1974), the 'intertwining number' for a real irreducible representation, i.e. the dimension of the space of the operators which commute with the representation, can be only 1, 2 or 4. The first case is usually realised in stationary bifurcation with one-dimensional zero eigenspace; the second one occurs in classical Hopf bifurcation, as we will briefly recall in the following. Finally, the third one—the 'quaternionic' case—is less known in bifurcation problems and, as suggested by Golubitsky, it would be interesting to construct some 'natural' example where this possibility occurs.

As a contribution in this direction, we will put here in explicit form a group theoretical situation where this type of 'quaternionic representation' is realised, and the form of a possible equation which exhibits a non-trivial 'quaternionic bifurcation'.

Let us start with Hopf bifurcation. An intrinsic and well known property of the Hopf bifurcation equation is its covariance with respect to time 'translations'. More precisely, by rescaling as usual the time variable t

$$t \to \tau = \omega t$$

in such a way that the problem is reduced to finding 2π periodic solutions (in the variable τ), the equation is covariant with respect to the group SO_2 (isomorphic to the circle S^1) of the 'translations' $\tau \to \tau + \tau'$ (mod 2π). The important point is that this property is shared by the (properly called) bifurcation equation in R^2, to which the problem reduces via the Lyapunov-Schmidt procedure, and which typically can be put in the form (under standard hypotheses, see, e.g., Chow and Hale 1982, Golubitsky 1983, Golubitsky and Stewart 1985, Sattinger 1979)

$$(\lambda - \lambda_0)Lv + (\omega - \omega_0)Mv + H(\lambda, \omega; v) = 0 \qquad v \in R^2 \tag{1}$$

where λ, ω are real parameters, L and M given matrices, and H the remaining nonlinear part, with $H(\lambda, \omega; 0)=0$. Of course, by a suitable choice of the basis, one can fix

$$M = J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{2}$$

where, as suggested by the notation, it is assumed that criticality occurs at $\lambda = \lambda_0$ and that the critical branches of eigenvalues $\sigma = \sigma(\lambda)$ satisfy $\sigma(\lambda_0) = \pm i\omega_0$.

A direct consequence of the SO_2 covariance of (1) is that the two matrices L and M have to commute with the standard real representation of SO_2

$$R(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \tag{3}$$

and this implies that L and M must be combinations of the two matrices J (equation (2)) and the identity I.

On the other hand, independent of SO_2 covariance, solvability of an equation such as equation (1) is ensured (as is well known, see especially Chow and Hale 1982) by the assumption that, for some vector $\hat{v} \in R^2$, the two vectors

$$L\hat{v} \qquad \text{and} \qquad M\hat{v} \tag{4}$$

are linearly independent. If this is true, the solution of (1) can be put (locally) in the form

$$\begin{aligned} &v = s\hat{v} && s \in R \\ &\lambda = \lambda(s) & \text{with}\quad & \lambda(0) = \lambda_0 \\ &\omega = \omega(s) && \omega(0) = \omega_0 \end{aligned} \tag{5}$$

which corresponds to a periodic solution with 'frequency' $\omega(s)$ of the original Hopf problem. Having already chosen $M = J$, the condition on vectors (4) becomes a condition on $L\hat{v}$, which in turn gives directly the usual 'transversality' assumption $(\partial\, \mathrm{Re}\, \sigma(\lambda)/\partial\lambda) \neq 0$ at $\lambda = \lambda_0$ on the critical branches of eigenvalues.

The above discussion shows that the very existence of non-zero solutions of the Hopf problem *is essentially due to a precise group theoretical property of the group* SO_2, i.e. to the rather 'exceptional' property of its real irreducible representation (3) of having *two* independent operators I and J which commute with the whole representation.

In order to construct now a concrete example of the group theoretical framework in which quaternionic bifurcation can occur, let us consider the simplest case of a real irreducible representation having four independent intertwining operators: this is provided by the representation of the unitary group SU_2 constructed as follows on the basis space R^4. Let

$$G_1 = \tfrac{1}{2}\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad G_2 = \tfrac{1}{2}\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$
$$G_3 = \tfrac{1}{2}\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \tag{6}$$

be the representatives of the Lie algebra generators of SU_2, obeying the SU_2 commutation rule

$$[G_i, G_j] = \varepsilon_{ijk} G_k \qquad i, j, k = 1, 2, 3; \tag{7}$$

then, the desired representation of SU_2, acting irreducibly on R^4, is given by

$$\exp(\phi_i G_i) = I \cos\frac{\phi}{2} + \left(\sin\frac{\phi}{2}\right)\frac{\phi_i G_i}{\phi} \equiv D(\phi_i) \tag{8}$$

where ϕ_i $(i = 1, 2, 3)$ are real parameters for SU_2, $\phi = (\phi_i\phi_i)^{1/2}$, and I is the identity on R^4. By complexification of the space, it is easily seen that, as expected, this representation is equivalent to the direct sum of two two-dimensional complex representations, each of which is equivalent to the 'canonical' representation of SU_2 as 2×2 unimodular unitary matrices.

Four independent operators commuting with the representation $D(\phi_i)$ (equation (8)) are now easily obtained:

$$Q_0 = I = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$$

and (9)

$$Q_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \qquad Q_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad Q_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

and it is seen that the operators Q_α $(\alpha = 0, 1, 2, 3)$ exhibit the quaternionic structure. It can be useful to note also that $\frac{1}{2}Q_i$ $(i = 1, 2, 3)$ generate another SU_2 group acting on R^4, and that both groups leave invariant the R^4-norm of vectors, in agreement with the property of the Lie algebra of SO_4 (the group of invariance of the R^4-norm) of being isomorphic to the direct sum of two commuting SU_2 algebras.

In this scheme, a typical bifurcation equation (analogous to (1)), written for vectors $u \in R^4$, covariant with respect to the representation (8) of SU_2, is expected to be of the form (under reasonable assumptions: see below for a precise statement)

$$\sum_{\alpha=0}^{3} (\lambda_\alpha - \lambda_\alpha^{(0)}) q_\alpha Q_\alpha u + H(\lambda; u) = 0 \tag{10}$$

where q_α are given real coefficients, λ_α four real variable parameters, and H the remaining nonlinear part, with $H(\lambda; 0) = 0$, commuting with the group action. The existence of non-zero solutions bifurcating from $\lambda_\alpha^{(0)} = 0$ is then guaranteed by the condition that

$$q_\alpha \neq 0 \qquad \forall \alpha = 0, 1, 2, 3.$$

In fact, if this is true, and $\hat{u}$ is any unit vector in R^4, the four vectors $Q_\alpha \hat{u}$ are linearly independent and then, by the implicit function theorem (cf in particular Cicogna and Degiovanni (1984) where the case of bifurcation with several parameters is explicitly treated) (10) has the non-trivial solution

$$u = s\hat{u}$$
$$\lambda_\alpha = \lambda_\alpha(s), \qquad \text{with } \lambda_\alpha(0) = \lambda_\alpha^{(0)} \qquad \alpha = 0, 1, 2, 3 \tag{11}$$

where s is a real parameter defined in a neighbourhood of zero. Note also that all other solutions constructed starting from different $\hat{u}$ are equivalent; in fact the orbit $D(\phi_i)\hat{u}$ of any vector $\hat{u}$ under the group action (8) describes the whole unit ball in R^4.

Summarising, the above results can be stated in the following fairly general form.

Theorem. Let $f = f(\lambda; u)$ be a regular (e.g. analytic) map f: $R^p \times R^4 \to R^4$, depending on p real parameters $\lambda \equiv (\lambda_1, \ldots, \lambda_p)$, satisfying $f(\lambda; 0) = 0$ and the 'covariance' property

$$f(\lambda; D(\phi_i)u) = D(\phi_i)f(\lambda; u) \tag{12}$$

with respect to the irreducible real representation (8) of the group SU_2. Then, the linearised part

$$f_u(\lambda; 0)$$

belongs to the real quaternionic field, i.e. it is a linear combination of matrices (9). If, in addition, in the case $p = 4$ one has $f_u(\lambda^{(0)}; 0) = 0$ for $\lambda^{(0)} \equiv (\lambda_0^{(0)}, \ldots, \lambda_3^{(0)})$, and the four operators

$$f_{u\lambda_\alpha}(\lambda^{(0)}; 0) \qquad \alpha = 0, 1, 2, 3$$

are linearly independent, then there exists a non-zero solution (together with the orbit of equivalent solutions obtained via the SU_2 action) of the equation $f(\lambda; u) = 0$, bifurcating at $\lambda_\alpha = \lambda_\alpha^{(0)}$, having the form (11). Each of these solutions destroys the original SU_2 symmetry, in the sense that there is no non-trivial subgroup of SU_2 which leaves it fixed.

A final nearly obvious remark: if—in more generality—one considers any bifurcation problem having SU_2 covariance, such that the kernel of the linearised part obtained by Fréchet differentiating the equation at the critical point (with usual regularity assumptions) is the basis space R^4 of the SU_2 representation (8), then this problem ultimately reduces (via standard Lyapunov-Schmidt projection) to the case covered by the above theorem. In fact, this is a consequence of the permanence of the covariance properties of the equations.

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